The vertex degree of a hierarchical long-range percolation graph

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Abstract

The use of percolation theory in statistical physics has long been recognized. In this letter, we consider percolation in the hierarchical lattice of order N where the probability of connection between two nodes separated by distance k is of the form $1 - e^{-\frac{\alpha}{\beta^k}}$, $\alpha \geq 0$ and $\beta > 0$. The vertex degrees of the resulting percolation graph are studied, which are consistent with the phase diagram of this model.

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1 Introduction and the model

The theory of percolation in the Euclidean lattice $\mathbb{Z}^d$ started with the work of Broadbent and Hammersley in 1957. The infinity of the space of sites (or nodes) and its geometry are principal features of this model (see e.g. [12, 19] for background). Some questions of percolation in other non-Euclidean infinite systems are formulated in [4]. The study of long-range percolation on $\mathbb{Z}^d$ goes back to [16] and leads to a number of interesting results in mathematical physics [1, 5, 6, 7, 8, 18, 20]. On the other hand, hierarchical structures have been used in applications in the physical, biological (in particular, genetic) and social sciences due to the multi-scale organization of many natural objects [3, 9, 14].

Recently, long-range percolation is studied on the hierarchical lattice Ω_N of order N (to be defined below), where classical methods for the usual lattice break down. The asymptotic long-range percolation on Ω_N is addressed in [11] for $N \to \infty$. The works [10, 13, 17] analyze the long-range percolation on Ω_N for finite N using different connection probabilities and methodologies. The contact process on Ω_N for fixed N has been investigated in [2]. In this letter, we focus on another basic quantity, the vertex degree, of long-range percolation graph on Ω_N for fixed N, which has not been treated yet. Our results are consistent with the phase transition of long-range percolation on Ω_N (see Section 2).

For an integer $N \geq 2$, we define the set

$$\Omega_N := \{\mathbf{x} = (x_1, x_2, \cdots) : x_i \in \{0, 1, \cdots, N-1\}, i = 1, 2, \cdots,$$
$$x_i \neq 0 \text{ only for finitely many } i\}, \quad (1)$$

and define a metric on it by

$$d(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \mathbf{x} = \mathbf{y}, \\ \max\{i : x_i \neq y_i\}, & \mathbf{x} \neq \mathbf{y}. \end{cases} \quad (2)$$

The pair (Ω_N, d) is called the hierarchical lattice of order N, which may be thought of as the set of leaves at the bottom of an infinite regular tree without a root, where the distance between two nodes is the number of levels (generations) from the bottom to their most recent common ancestor, see Figure 1.

Such a distance d satisfies the strong triangle inequality

$$d(\mathbf{x}, \mathbf{y}) \leq \max\{d(\mathbf{x}, \mathbf{z}), d(\mathbf{z}, \mathbf{y})\}, \quad (3)$$

for any triple $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \Omega_N$. Hence, (Ω_N, d) is an ultrametric (or non-Archimedean) space [15]. From its ultrametricity, it is easy to see that for every $\mathbf{x} \in \Omega_N$ there are $(N-1)N^{k-1}$ nodes at distance k from it.

Now consider a long-range percolation on Ω_N. For each $k \geq 1$, the probability of connection between $\mathbf{x}$ and $\mathbf{y}$ such that $d(\mathbf{x}, \mathbf{y}) = k$ is given by

$$p_k = 1 - e^{-\frac{\alpha}{\beta^k}}, \quad (4)$$

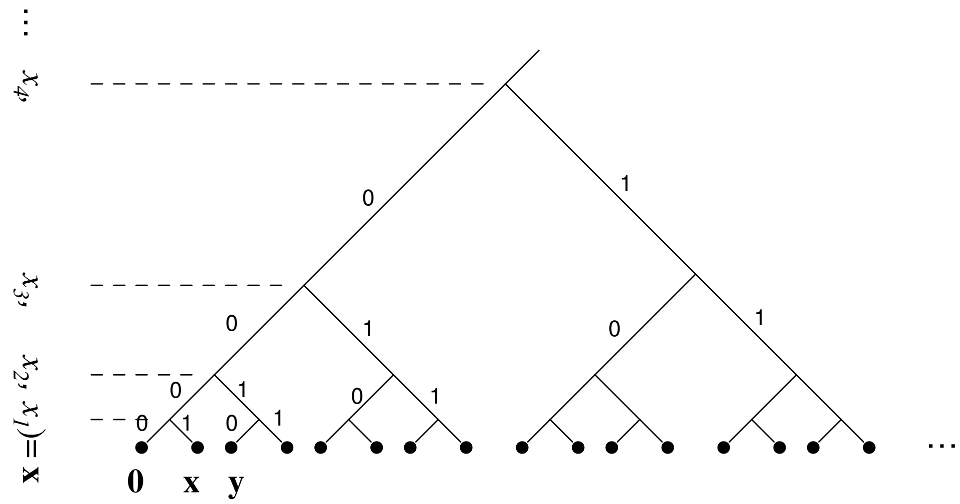

Figure 1: An illustration of hierarchical lattice Ω_2 of order 2. The distances between three nodes $\mathbf{0} = (0, 0, 0, \cdots)$, $\mathbf{x} = (1, 0, 0, \cdots)$ and $\mathbf{y} = (0, 1, 0, \cdots)$ are $d(\mathbf{0}, \mathbf{x}) = 1$ and $d(\mathbf{0}, \mathbf{y}) = d(\mathbf{x}, \mathbf{y}) = 2$.

where $0 \le \alpha < \infty$ and $0 < \beta < \infty$, all connections being independent. Koval et al. [13] show the uniqueness of the infinite component as well as phase diagram of this long-range percolation model with respect to the parameters α and β (see Section 2 below). In this letter, we want to explore the vertex degree of the resulting percolation graph.

The rest of the letter is organized as follows. In section 2, we present the vertex degree and some related results and the proofs are given in Section 3. We draw a brief conclusion in Section 4.

2 The results

We first present some known phase transition results for the above percolation model.

Let $|S|$ be the size of a set S. The connected component containing the node $\mathbf{x}$ is denoted by $C(\mathbf{x})$. Since, for every $\mathbf{x} \in \Omega_N$, $|C(\mathbf{x})|$ has the same distribution, it suffices to consider only $|C(\mathbf{0})|$. The percolation probability is defined as

$$\theta(\alpha, \beta) := P(|C(\mathbf{0})| = \infty), \tag{5}$$

and the critical percolation value is defined as

$$\alpha_c(\beta) := \inf\{\alpha \geq 0 : \theta(\alpha, \beta) > 0\}. \tag{6}$$

The phase diagram is established in the following result.

Theorem 1.([13])

(i) $\alpha_c(\beta) = 0$ *for* $\beta \leq N$*;*

(ii) $0 < \alpha_c(\beta) < \infty$ *for* $N < \beta < N^2$*;*

(iii) $\alpha_c(\beta) = \infty$ *for* $\beta \geq N^2$.

For $\mathbf{x} \in \Omega_N$, denote by $D_{\mathbf{x}}$ the degree of node $\mathbf{x}$ in the resulting percolation graph. Since $D_{\mathbf{x}}$ has the same distribution for every $\mathbf{x} \in \Omega_N$, we may study $D_{\mathbf{0}}$ instead of $D_{\mathbf{x}}$. The main result of this letter is the following

Theorem 2. *Let* $\mathbb{N}$ *be the non-negative integers including 0.*

(i) If $\beta \leq N$ *and* $\alpha > 0$, *then* $P(D_{\mathbf{0}} = \infty) = 1$*;*

(ii) If $\beta > N$ *and* $\ell := \min\{k \in \mathbb{N} : \alpha \leq \beta^{k+1}\}$, *then*

$$\frac{N^\ell - 1}{2} + \frac{\alpha(N-1)N^\ell}{2(\beta - N)\beta^\ell} \leq ED_{\mathbf{0}} \leq \frac{\alpha(N-1)}{\beta - N}. \tag{7}$$

In particular, if $\ell = 0$, *then* $\frac{\alpha(N-1)}{2(\beta-N)} \leq ED_{\mathbf{0}} \leq \frac{\alpha(N-1)}{\beta-N}$.

From Theorem 2 (i) we know that $D_{\mathbf{0}}$ is almost surely infinite for $\beta \leq N$, which immediately yields the result in Theorem 1 (i). Theorem 2 (ii) implies that $D_{\mathbf{0}}$ is almost surely bounded for $\beta > N$, and what's more, when α is small enough, $\ell = 0$ and thus the expectation of $D_{\mathbf{0}}$ approaches 0. This implies the results in Theorem 1 (ii) and (iii) that the critical value $\alpha_c(\beta) > 0$ when $\beta > N$.

3 Proof of Theorem 2

Proof of (i). Let E_k be the event that $\mathbf{0}$ connects by an edge to at least one node at distance k. Therefore, by (4) and the fact that there are $(N-1)N^{k-1}$ nodes at distance k from the origin $\mathbf{0}$. we have

$$P(E_k) = 1 - (1 - p_k)^{(N-1)N^{k-1}} = 1 - e^{-\frac{\alpha}{\beta^k}(N-1)N^{k-1}}. \tag{8}$$

If $\beta \leq N$, the sum $\sum_{k=1}^{\infty} P(E_k)$ diverges for any $\alpha > 0$. Since the events $\{E_k\}_{k\geq 1}$ are independent, it then follows from the Borel-Cantelli lemma that infinitely many of the event E_k occur with probability 1. Hence $P(D_{\mathbf{0}} = \infty) = 1$. $\square$

Proof of (ii). To begin with, we obtain the expected vertex degree

$$ED_{\mathbf{0}} = \sum_{k=1}^{\infty}(N-1)N^{k-1}p_k = \sum_{k=1}^{\infty}(N-1)N^{k-1}\left(1 - e^{-\frac{\alpha}{\beta^k}}\right). \tag{9}$$

Since $1 - e^{-x} \leq x$, we obtain from (9) that

$$ED_{\mathbf{0}} \leq \frac{\alpha(N-1)}{N}\sum_{k=1}^{\infty}\left(\frac{N}{\beta}\right)^k = \frac{\alpha(N-1)}{\beta - N} \tag{10}$$

for $\beta > N$.

On the other hand, using the bound $1 - e^{-x} \geq \frac{\min\{x,1\}}{2}$, we get

$$\begin{aligned} ED_{\mathbf{0}} &\geq \frac{N-1}{2N}\sum_{k=1}^{\infty}\min\left\{\frac{\alpha N^k}{\beta^k}, N^k\right\} \\ &= \frac{N-1}{2N}\left(\sum_{k=1}^{\ell} N^k + \sum_{k=\ell+1}^{\infty}\frac{\alpha N^k}{\beta^k}\right) \\ &= \frac{N^\ell - 1}{2} + \frac{\alpha(N-1)N^\ell}{2(\beta - N)\beta^\ell}, \end{aligned} \tag{11}$$

by (9) and the definition of ℓ. The proof is then complete. $\square$

4 Conclusion

The use of percolation theory in statistical physics has long been recognized. In this letter, we characterize the vertex degree of a hierarchical long-range percolation graph. The results are consistent with the phase transition in percolation. The graph distance and diameter of the percolation graph are interesting future work.

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